DO NOW: **SHOW ALL WORK IN YOUR NOTEBOOK!**

Find:

1. $\int \sin(x)\cos^6(x)dx$

2. $\int \tan(x)\sec^2(x)dx$ **Work this problem out in two different ways**

3. $\int \sin(2x)\cos(x)dx$

1. $\int \sin x \cos^6 x \, dx = -\int u^6 du = -\frac{u^7}{7} + C = \boxed{-\frac{1}{7}\cos^7(x) + C}$

$u = \cos x$
$du = -\sin x \, dx$

2. $\int \tan x \sec^2 x \, dx = \int u \, du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\tan^2 x + C}$

$u = \tan x$
$du = \sec^2 x \, dx$

$\int \tan x \sec^2 x \, dx = \int u \, du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\sec^2 x + C}$

$u = \sec x$
$du = \tan x \sec x \, dx$

Equivalent answers!!!
$(\tan^2 x + 1 = \sec^2 x)$

3. $\int \sin(2x) \cos(x) \, dx$

$\int 2\sin x \cos x \cdot \cos x \, dx$

$\int 2\sin x \cos^2 x \, dx = -2\int u^2 du = -\frac{2}{3}u^3 + C = \boxed{-\frac{2}{3}\cos^3 x + C}$

$u = \cos x$ $du = -\sin x \, dx$

INTEGRALS INVOLVING POWERS OF SINE AND COSINE

We'll study techniques for evaluating integrals of the form

$$\int \sin^m(x)\cos^n(x)dx \quad \text{and} \quad \int \sec^m(x)\tan^n(x)dx$$

whether either *m* or *n* is a positive integer. The idea is to transform the integrand in a way that will allow you to use the same reasoning as in theO NOW.

Before we discuss the technique we need to remember the following Trig. Identities.

$$\sin^2(x)+\cos^2(x)=1$$

$$\sin^2(x)=\frac{1-\cos(2x)}{2}$$

$$\cos^2(x)=\frac{1+\cos(2x)}{2}$$

<u>Guidelines</u> for Evaluating Integrals of the form $\int \sin^m(x)\cos^n(x)dx$

1. If the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosines. Then expand and integrate. **See Example #1**

2. If the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sines. Then expand and integrate. **See Example #2**

3. If the power of both sine and cosine are even and nonnegative, make repeat use of the identities

$$\sin^2(x)=\frac{1-\cos(2x)}{2} \quad \text{and} \quad \cos^2(x)=\frac{1+\cos(2x)}{2}$$

See Example #3

Example #1: **SHOW ALL WORK IN YOUR NOTEBOOK!**

Find $\int \sin^5(x)\cos^4(x)dx = \int \sin x \cdot \sin^4 x \cos^4 x \, dx =$

$= \int \sin x (1-\cos^2 x)^2 \cos^4 x \, dx = \int \sin x (1 - 2\cos^2 x + \cos^4 x)\cos^4 x \, dx$

$= \int \sin x (\cos^4 x - 2\cos^6 x + \cos^8 x) dx$

$u = \cos x$

$du = -\sin x \, dx$

$-\int u^4 - 2u^6 + u^8 du = -\frac{1}{5}u^5 + \frac{2}{7}u^7 - \frac{1}{9}u^9 + C$

$$\boxed{-\frac{1}{5}\cos^5 x + \frac{2}{7}\cos^7 x - \frac{1}{9}\cos^9 x + C}$$

Example #2:

Find $\int \frac{\cos^3(x)}{\sqrt[3]{\sin(x)}}\,dx = \int \frac{\cos x \cdot \cos^2 x}{\sqrt[3]{\sin x}}\,dx = \int \frac{\cos x\,(1-\sin^2 x)}{(\sin x)^{1/3}}\,dx$

$u = \sin x$

$du = \cos x\,dx$

$$\int \frac{1-u^2}{u^{1/3}}\,du = \int u^{-1/3} - u^{5/3}\,du = \frac{3}{2}u^{2/3} - \frac{3}{8}u^{8/3} + C$$

$$\frac{3}{2}\sin^{2/3} x - \frac{3}{8}\sin^{8/3} x + C$$

$$\boxed{\frac{3}{2}\sqrt[3]{\sin^2 x} - \frac{3}{8}\sin^2 x\sqrt[3]{\sin^2 x} + C}$$

Example #3:

Find $\int \cos^4(x)\sin^2(x)dx$ $= \bigstar$

$$\cos^4 x \sin^2 x = \left(\frac{1+\cos(2x)}{2}\right)^2\left(\frac{1-\cos(2x)}{2}\right)$$

$$\frac{(1+2\cos(2x)+\cos^2(2x))(1-\cos(2x))}{8}$$

$$\frac{1+2\cos(2x)+\cos^2(2x)-\cos(2x)-2\cos^2(2x)-\cos^3(2x)}{8}$$

$$\frac{1+\cos(2x)-\cos^2(2x)-\cos^3(2x)}{8}$$

$$\underbrace{\frac{1}{8}}+\underbrace{\frac{1}{8}\cos(2x)}-\underbrace{\frac{1}{16}(1+\cos(4x))}-\underbrace{\frac{1}{8}\cos(2x)(1-\sin^2(2x))}$$

$$\bigstar = \int \frac{1}{8}dx + \frac{1}{8}\int \cos(2x)\,dx - \frac{1}{16}\int 1+\cos(4x)\,dx - \frac{1}{8}\int \cos(2x)(1-\sin^2(2x))\,dx$$

$$= \frac{1}{8}x + \frac{1}{16}\sin(2x) - \frac{1}{16}x - \frac{1}{64}\sin(4x) - \frac{1}{8}\int \cos(2x)(1-\sin^2(2x))\,dx$$

$u = \sin(2x)$

$du = 2\cos(2x)dx$

$$-\frac{1}{16}\int 1-u^2\,du = -\frac{1}{16}u + \frac{1}{48}u^3$$

$$= -\frac{1}{16}\sin(2x) + \frac{1}{48}\sin^3(2x)$$

$$= \frac{1}{8}x + \cancel{\frac{1}{16}\sin(2x)} - \frac{1}{16}x - \frac{1}{64}\sin(4x) - \cancel{\frac{1}{16}\sin(2x)} + \frac{1}{48}\sin^3(2x) + C$$

$$= \boxed{\frac{1}{16}x - \frac{1}{64}\sin(4x) + \frac{1}{48}\sin^3(2x) + C}$$

PLEASE refer to your textbook, pg. 537, to see the WALLIS'S FORMULAS.

INTEGRALS INVOLVING POWERS OF SECANT AND TANGENT

Guidelines for Evaluating Integrals of the form $\int \sec^m(x)\tan^n(x)dx$

1. If the power of the secant is even and positive, save a secant-squared factor and convert the remaining factors to tangents. Then expand and integrate.
   See Example #4

2. If the power of the tangent is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Then expand and integrate.
   See Example #5

3. If there no are secant factors and the power of the tangent is even and positive, convert a tangent-squared factor to a secant-squared factor. Then expand and integrate. **See Example #6**

4. If the integral is of the form $\int \sec^m(x)dx$, where *m* is odd and positive, use Integration by Parts. **See Example #7**

5. If none of the first four guidelines applies, try converting to sines and cosines.
   See Example #8

Example #4: **SHOW ALL WORK IN YOUR NOTEBOOK!**

Find $\int \sec^4(2x)\tan^3(2x)dx$

$$\int \sec^4(2x)\tan^3(2x)dx = \int \sec^2(2x)\cdot\sec^2(2x)\tan^3(2x)\,dx$$

$$\int \sec^2(2x)\left[1+\tan^2(2x)\right]\tan^3(2x)\,dx$$

$$\int \sec^2(2x)\left[\tan^3(2x)+\tan^5(2x)\right]dx$$

$u = \tan(2x)$

$du = \sec^2(2x)\cdot 2\,dx = 2\sec^2(2x)\,dx$

$$\frac{1}{2}\int u^3+u^5\,du = \frac{1}{2}\left(\frac{u^4}{4}+\frac{u^6}{6}\right)+C = \frac{u^4}{8}+\frac{u^6}{12}+C$$

$$\boxed{\frac{1}{8}\tan^4(2x)+\frac{1}{12}\tan^6(2x)+C}$$

Example #5:

Find $\int \frac{\tan^3(3x)}{\sqrt{\sec(3x)}}\,dx = \int \tan^3(3x)\sec^{1/2}(3x)\,dx$

$$\int \tan(3x)\sec(3x)\left[\tan^2(3x)\sec^{-1/2}(3x)\right]dx$$

$$\int \tan(3x)\sec(3x)\left[(\sec^2(3x)-1)\sec^{-1/2}(3x)\right]dx$$

$$\int \tan(3x)\sec(3x)\left[\sec^{3/2}(3x)-\sec^{-1/2}(3x)\right]dx$$

$u = \sec(3x)$

$du = \tan(3x)\sec(3x)\cdot 3\,dx$

$$\frac{1}{3}\int u^{3/2}-u^{-1/2}\,du = \frac{1}{3}\left(\frac{2}{5}u^{5/2}-2u^{1/2}\right)+C$$

$$\frac{2}{15}\sec^{5/2}(3x)-\frac{2}{3}\sec^{1/2}(3x)+C$$

$$\boxed{\frac{2}{15}\sec^2(3x)\sqrt{\sec(3x)}-\frac{2}{3}\sqrt{\sec(3x)}+C}$$

Example #6:

Find $\int \tan^4(2x+1)dx$ $= \int \tan^2(2x+1)\tan^2(2x+1)\,dx$

$$\int \tan^2(2x+1)\left[\sec^2(2x+1)-1\right]dx$$

$$\int \tan^2(2x+1)\sec^2(2x+1)dx - \int \tan^2(2x+1)dx$$

$$\int \tan^2(2x+1)\sec^2(2x+1)\,dx - \int \sec^2(2x+1)-1\,dx$$

$$\int \tan^2(2x+1)\sec^2(2x+1)\,dx - \int \sec^2(2x+1)\,dx + \int dx$$

$u = \tan(2x+1)$

$du = 2\sec^2(2x+1)$

$$\frac{1}{2}\int u^2 du = \frac{1}{6}u^3 + c$$

$$\boxed{\frac{1}{6}\tan^3(2x+1) - \frac{1}{2}\tan(2x+1) + x + C}$$

Example #7:

Find $\int \sec^3(x)dx = \int \sec x \cdot \sec^2 x \, dx$

$u = \sec x \qquad dv = \sec^2 x \, dx$

$du = \sec x \tan x \, dx \qquad v = \tan x$

$\sec x \tan x - \int \sec x \tan^2 x \, dx$

$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$

$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x + \int \sec x \, dx$

$2\int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$

$$\int \sec x \, dx = \int \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x} dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x} dx = \int \frac{1}{u} du$$

$$= \ln|u| + C$$

$$= \ln|\tan x + \sec x| + C$$

$u = \tan x + \sec x$

$du = (\sec^2 x + \sec x \tan x) dx$

$$\boxed{\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\tan x + \sec x| + C}$$

Example #8:

Find $\int \frac{\sec(x)}{\tan^2(x)}dx$

$$\frac{\frac{1}{\cos x}}{\frac{\sin^2 x}{\cos^2 x}} = \frac{1}{\cos x} \cdot \frac{\cos^2 x}{\sin^2 x} = \frac{\cos x}{\sin^2 x}$$

$$\int \frac{\cos x}{\sin^2 x}dx = \int u^{-2}du = -u^{-1}+C$$

$$= -\frac{1}{u}+C$$

$$= -\frac{1}{\sin x}+C = \boxed{-\csc x+C}$$

$u = \sin x$

$du = \cos x\,dx$

INTEGRALS INVOLVING SINE – COSINE PRODUCTS WITH DIFFERENT ANGLES

REMINDER OF Product to Sum Identities

$$\sin(mx)\sin(nx) = \frac{1}{2}\left(\cos\left[(m-n)x\right] - \cos\left[(m+n)x\right]\right)$$

$$\sin(mx)\cos(nx) = \frac{1}{2}\left(\sin\left[(m-n)x\right] + \sin\left[(m+n)x\right]\right)$$

$$\cos(mx)\cos(nx) = \frac{1}{2}\left(\cos\left[(m-n)x\right] + \cos\left[(m+n)x\right]\right)$$

Example #9:

Find $\int \sin(8x)\cos(3x)\,dx$

$\sin(8x)\cos(3x) =$

$= \frac{1}{2}\left(\sin(5x) + \sin(11x)\right)$

$\frac{1}{2}\int \sin(5x)\,dx + \frac{1}{2}\int \sin(11x)\,dx$

$$\boxed{-\frac{1}{10}\cos(5x) - \frac{1}{22}\cos(11x) + C}$$